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On Multi-Order Fractional Differential Operators in the Unit Disk

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Abstract. In this article, we generalize fractional operators (differential and integral) in the unit disk. These operators are generalized the Srivastava-Owa operators. Geometric properties are studied and the advantages of these operators are discussed. As an application, we impose a method, involving a memory formalism of the Beer-Lambert equation based on a new generalized fractional differential operator. We give solutions in terms of the multi-index Mittag-Leffler function. In addition, we sanctify the out come from a stochastic standpoint. We utilize the generalized Wright function to obtain the analytic formula of solutions.

1. Introduction

The Beer-Lambert equation or the Beer-Lambert law is commonly utilized in spectroscopy to obtain the absorption coefficient of non-scattering media from continuous wave measurements. For scattering media, a relaxed Beer-Lambert law has been inserted in [1] and is generally employed in the field of near-infrared tissue spectroscopy. Later, the equation has been generalized and modified by many authors and researches [2 − 4]. Recently, Patterson et. al., demonstrated the failure of the Beer-Lambert law due to multiple scattering [5]. Saitoh et. al., applied the Beer-Lambert law to measure the leaf area index and they showed the advantages of using this method [6]. Lapuerta et. al., imposed a new idea for the determination of the number of primary particles composing soot agglomerates emitted from diesel engines as well as their individual fractal dimension. This method is completely based on the Beer-Lambert law [7].

Newly, Tramontana et. al., introduced a different approach, including a memory formalism in the classical Beer-Lambert law through fractional calculus modeling [8]. The fractional differential operators are taken in sense of the Caputo fractional derivative of one parameter. Later, Tramontana et. al., developed the Beer-Lambert law, by applying a Wright type function, to describe the probability of photon transmission in random media. They found the analytic form of the photon mean-free-path related to such Wright formula of extinction [9].

Fractional calculus of any positive order can be considered as a branch of mathematical physics, associated with differential equations and inclusions, integral equations and integro-differential equations, where integrals are of convolution form with weak singular kernels of power law type [10 − 12].

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In this work, we propose a process, involving a memory formalism of the Beer-Lambert equation based on generalized fractional differential operator. The fractional calculus is assumed in sense of the multi-order parameters. We formulate the solution in terms of the multi-index Mittag-Leffler function. Furthermore, we sanctify the out come from a stochastic standpoint. We use the generalized Wright function to yield the analytic formula of the solution. The advantages of this method are discussed.

2. Fractional Calculus

This section deals with some preliminaries and notations regarding the fractional calculus. In [13], Srivastava and Owa, defined and studied fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ for analytic functions

Definition 2.1. *The fractional derivative of order β is read, for a function $h(z)$ by*

$$D_z^\beta h(z) := \frac{1}{\Gamma(1-\beta)}\frac{d}{dz}\int_0^z \frac{h(\zeta)}{(z-\zeta)^\beta}d\zeta; \quad 0 \le \beta < 1,$$

such that $h(z)$ is analytic in simply-connected region in $\mathbb{C}$. This region is contained the origin and the multiplicity of $(z-\zeta)^{-\beta}$ is removed by requiring $\log(z-\zeta)$ to be real when$(z-\zeta) > 0$. Furthermore, for $n \le \beta < n+1$, the fractional differential operator is formulated by

$$D_z^\beta h(z) = \frac{d^n}{dz^n}D_z^{\beta-n}h(z), \quad n \in \mathbb{N}.$$

Definition 2.2. *The fractional integral of order α is read, for a function $h(z)$, by*

$$I_z^\beta h(z) := \frac{1}{\Gamma(\beta)}\int_0^z h(\zeta)(z-\zeta)^{\beta-1}d\zeta; \quad \beta > 0,$$

such that $h(z)$ is analytic in simply-connected region in $\mathbb{C}$ involving the origin. In addition, the multiplicity of $(z-\zeta)^{\beta-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 2.3. [13]

$$D_z^\beta z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\beta+1)}z^{\mu-\beta}, \ \ \mu > -1; \ 0 \le \beta < 1$$

and

$$I_z^\beta z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\beta+1)}z^{\mu+\beta}, \ \ \mu > -1; \ \beta > 0.$$

Recently, the Srivastava-Owa operators are generalized for two parameters and modified in [14, 15] respectively. Other observations, including these operators can be found in [16 − 19].

The Fox-Wright function ${}_p\Psi_q[z]$ defined by

$$\begin{aligned} {}_p\Psi_q\left[\begin{array}{ll} (a_1, A_1), ..., (a_p, A_p); & \\ & z \\ (b_1, B_1), ..., (b_q, B_q); & \end{array}\right] &= {}_p\Psi_q[(a_j, A_j)_{1,p}; (b_j, B_j)_{1,q}; z] \\ &:= \sum_{n=0}^{\infty} \frac{\Gamma(a_1+nA_1)...\Gamma(a_q+nA_p)}{\Gamma(b_1+nB_1)...\Gamma(b_q+nB_q)}\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{z^n}{n!} \end{aligned}$$

where $a_j, b_j \in \mathbb{R}$, $A_j > 0$ for all $j = 1, ..., p$, $B_j > 0$ for all $j = 1, ..., q$ and $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0$ for $|z| < 1$.

The usage of the Fox-Wright function is sufficiently interesting. However, it connected with the Dziok-Srivastava multi-index convolution operator (see [20] and [21]) as well as the Srivastava-Wright operator (see [22] and [23]).

Now, we let $A_j = B_j = \frac{1}{\alpha_j}$, $\alpha_j > 0$ and $a_j = b_j$, then for analytic function in the open unit disk U,

$$f(z) = \sum_{n=0}^{\infty} \phi_n z^n, \quad z \in U,$$

we define the fractional operators

$$\mathfrak{D}_z^{\alpha_j, a_j} f(z) = \sum_{n=1}^{\infty} \phi_n \frac{\prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n-1}{\alpha_j})} z^{n-1}, \tag{1}$$

$$\mathfrak{I}_z^{\alpha_j, a_j} f(z) = \sum_{n=0}^{\infty} \phi_n \frac{\prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n+1}{\alpha_j})} z^{n+1}. \tag{2}$$

We call them, Srivastava-Owa fractional multi-order differentiation and integration operators respectively. We note that

$$\mathfrak{D}_z^{\alpha_j, a_j} \mathfrak{I}_z^{\alpha_j, a_j} f(z) = f(z) = \sum_{n=0}^{\infty} \phi_n z^n, \quad z \in U.$$

It is well known that

$$\mathfrak{E}_{1/\alpha_j, a_j}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j})} = {}_1\Psi_p \left[\begin{array}{cc} (1,1); & \\ & z \\ (a_1, \frac{1}{\alpha_1}), ..., (a_p, \frac{1}{\alpha_p}); & \end{array} \right],$$

where $\mathfrak{E}_{1/\alpha_j, a_j}(z)$ is the multi-index Mittag-Leffler functions [24]. One can realize that

$$\mathfrak{D}_z^{\alpha_j, a_j} \mathfrak{E}_{1/\alpha_j, a_j}(\lambda z) = \lambda \mathfrak{E}_{1/\alpha_j, a_j}(\lambda z), \quad \lambda \neq 0. \tag{3}$$

Recently, the advantages of the Mittag-Leffler function and its popularity increased significantly due to its important role in applications and fractional of arbitrary orders related differential and integral equations of fractional order, solutions to the problems of control theory, fractional viscoelastic models, diffusion theory, continuum mechanics and fractals [25, 26]. Latterly, numerical routines for Mittag-Leffler functions have been introduced, e.g., by Freed et al. [27], Gorenflo et al. [28] (with MATHEMATICA), Podlubny [29] (with MATLAB), Seybold and Hilfer [30].

3. Fractional Beer-Lambert Equation

In this section, we generalize the Beer-Lambert equation by employing operator (1). Consider the fractional equation

$$\mathfrak{D}_z^{\alpha_j, a_j} \Upsilon(z) = -\epsilon \Upsilon(z), \quad z \in U \tag{4}$$

subject to the initial condition

$$\Upsilon(0) = \Upsilon_0.$$

By applying (3), with $\lambda = -\epsilon = \Upsilon_0$, we may conclude that Eq.(4) satisfying the Mittag-Leffler function and

$$\Upsilon(z) = \Upsilon_0 \mathfrak{E}_{1/\alpha_j, a_j}(-\epsilon z). \tag{5}$$

Next our aim is to find solutions for (4), utilizing fractional probability of extinction and the Fox-Wright functions. In [8], the authors generalized the probability of extinction, by applying the fractional Poisson process of one parameter as follows:

$$P_\mu(k,z) = \frac{(\nu z)^k}{k!} \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} \frac{(-\nu z^\mu)^n}{\Gamma(\mu(n+k)+1)}$$

and the probability of transmission

$$P_\mu(0,z) = \sum_{n=0}^{\infty} \frac{(-\nu z^\mu)^n}{\Gamma(\mu n+1)} = \mathfrak{E}_{\mu,1}(-\nu z^\mu).$$

By using the concept of generalized operators (1) and (2), we my introduce the multi-order fractional probability of extinction as follows:

$$P_{1/\alpha_j,a_j}(k,z) = \frac{(\nu z)^k}{k!} \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} \frac{(-\nu z^\mu)^n}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n+k}{\alpha_j})}$$

and the probability of transmission

$$P_{1/\alpha_j,a_j}(0,z) = \sum_{n=0}^{\infty} \frac{(-\nu z^\mu)^n}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j})} = \mathfrak{E}_{1/\alpha_j,a_j}(-\nu z^\mu),$$

where ν is a parameter, $\alpha_j > 0$ and $a_j \in \mathbb{R}$. Clearly we have a multi-order of the classical Poisson prediction with a class of analytic functions, involving the exponential.

Distinctly

$$\Upsilon(z) = \Upsilon_0 \mathfrak{E}_{1/\alpha_j,a_j}(-\epsilon z^\mu) \tag{6}$$

is a solution for (4). This solution assumes that fractional modeling is a good tool to characterize, without empirical assumptions, complex power law attitude. We impose, by using a stochastic approach from the space fractional Poisson process of multi-order, the same result of the generalized fractional Beer-Lambert equation. The Mittag-Leffler is a interesting distribution and mostly decays slower than the exponential one. This manner yields an infinite mean-free-path distribution, which leads an infinite mean distance between obstacles. This result looks to be realistic incases of really diluted gases only. Beginning from this idea, and supposing that multi-order Wright functions are directly connected to anomalous diffusion processes, the next progressing is to find a solution of (4).

Let $Z_1, Z_2, ...$ be independent identically distributed variables with $\Re(Z_i) > 0$ satisfying the following distribution

$$\Theta(z) := P(\Re Z \le \Re z) = 1 - W_{1/\alpha_j,a_j}(-\epsilon z^{\frac{1}{\alpha_1}+...+\frac{1}{\alpha_p}}), \quad \alpha_i > 0, \tag{7}$$

where ρ is a constant and $W_{1/\alpha_j,a_j}$ is the multi-order Wright function

$$W_{1/\alpha_j,a_j}(-z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n! \prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j})}.$$

Note that for $p = 1$, $\frac{1}{\alpha} = 1 - \mu$, $a = \mu$, the last assertion implies a result obtained in [9]. The probability density function of (7) is given by

$$\theta(z) = \frac{d}{dz}\Theta(z), \quad \text{with} \quad \int_{\partial U} \theta(z)dz = 1, \, z \in U.$$

Let

$$\Lambda(z) = 1 - \Theta(z) := \overline{W}_{1/\alpha_j,a_j}(-\epsilon z^{\frac{1}{\alpha_1}+...+\frac{1}{\alpha_p}}), \tag{8}$$

we have the following result.

Theorem 3.1. *Assume* $\Theta(z)$ *and* $\Lambda(z)$ *as in* (7) *and* (8) *respectively. Then Eq.*(4) *has a solution in terms of* $\overline{W}_{1/\alpha_j,a_j}$.

Proof. Subsidiary [31], the probability can be viewed as the free-path-distribution law. It is read by

$$dP(z) = |\frac{d\Lambda}{dz}|dz.$$

From (8), we obtain

$$\begin{aligned}
\frac{d\Lambda}{dz} &= \frac{d}{dz}\Big(\sum_{n=0}^{\infty} \frac{(-1)^n \epsilon^n z^{n(\frac{1}{\alpha_1}+...+\frac{1}{\alpha_p})}}{n! \prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j})}\Big) \\
&= \Big(\frac{1}{\alpha_1} + ... + \frac{1}{\alpha_p}\Big)\Big(\sum_{n=0}^{\infty} \frac{(-1)^n \epsilon^n z^{n(\frac{1}{\alpha_1}+...+\frac{1}{\alpha_p})-1}}{(n-1)! \prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j})}\Big) \\
&= z^{-1}\Big(\frac{1}{\alpha_1} + ... + \frac{1}{\alpha_p}\Big)\Big(\sum_{n=0}^{\infty} \frac{(-1)^n \epsilon^n z^{n(\frac{1}{\alpha_1}+...+\frac{1}{\alpha_p})}}{\Gamma(n)\Gamma(n+1) \prod_{j=2}^{p} \Gamma(a_j + \frac{n}{\alpha_j})}\Big), \quad \text{for some } a_j = \alpha_j = 1 \\
&:= \frac{z^{-1}}{\alpha} \sum_{n=0}^{\infty} \omega(n) \frac{(-1)^n \epsilon^n z^{\frac{n}{\alpha}}}{n!}, \quad z \neq 0 \\
&= \frac{z^{-1}}{\alpha} \overline{W}_{1/\alpha_j,a_j}(-\epsilon z^{\frac{1}{\alpha}}),
\end{aligned}$$

where

$$\frac{1}{\alpha} := \frac{1}{\alpha_1} + ... + \frac{1}{\alpha_p}$$

and

$$\omega(n) := \prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j}).$$

Thus, we receive

$$P(z) = \int_{\partial U} |\frac{z^{-1}}{\alpha} \overline{W}_{1/\alpha_j, a_j}(-\epsilon z^{\frac{1}{\alpha}})| dz$$

or equivalents to

$$P(x) = \int_0^\infty \frac{x^{-1}}{\alpha} \overline{W}_{1/\alpha_j, a_j}(-\epsilon x^{\frac{1}{\alpha}}) dx, \quad x \geq 0.$$

Now in view of the Ramanujan Master theorem, we get

$$\int_0^\infty \frac{x^{-1}}{\alpha} \overline{W}_{1/\alpha_j, a_j}(-\epsilon x^{\frac{1}{\alpha}}) dx = \frac{\Gamma(0)\omega(0)}{\alpha}, \quad \alpha > 0, \ \omega(0) \neq 0.$$

Hence we can formulate a solution of (4) as follows

$$\Upsilon(z) = \Upsilon_0 \overline{W}_{1/\alpha_j, a_j}(-\epsilon z^{\frac{1}{\alpha}}).$$

□

4. Geometric Properties

In this section, we discus some geometric properties of the fractional operators (1) and (2). One of the major branches of complex analysis is univalent function theory: the study of one-to-one analytic functions. A domain E of the complex plane is said to be convex if and only if the line segment joining any two points of E lies entirely in E : An analytic, univalent function f in the unit disk U mapping the unit disk onto some convex domain is called a convex function.

Let $\mathcal{A}$ be the class of functions $h(z)$ normalized by $h(z) = z + \sum_{n=2}^\infty \phi_n z^n, \quad z \in U$. In addition, let $\mathcal{S}$ and C be the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, univalent and convex in U. It is clear that; if the function $h(z)$ is in the class $\mathcal{S}$, then

$$|\phi_n| \leq n, \quad n \in \mathbb{N}\backslash\{1\}. \tag{9}$$

Moreover, if the function $h(z)$ is in the class C, then

$$|\phi_n| \leq 1, \quad n \in \mathbb{N}. \tag{10}$$

Theorem 4.1. *Let the function h be in $\mathcal{S}$ (the class of univalent function). Then*

$$|\mathfrak{D}_z^{\alpha_j, a_j} h(z)| \leq \left({}_{p+1}\Psi_p \left[\begin{array}{c} (a_1, \frac{1}{\alpha_1}), ..., (a_p, \frac{1}{\alpha_p}), (1,1); \\ (a_1 - \frac{1}{\alpha_1}, \frac{1}{\alpha_1}), ..., (a_p - \frac{1}{\alpha_p}, \frac{1}{\alpha_p}); \end{array} |z| \right] \right)'.$$

Proof. Suppose that the function $h(z) \in \mathcal{S}$ is given by $h(z) = z + \sum_{n=2}^\infty \phi_n z^n$. Then we have

$$\begin{aligned} \mathfrak{D}_z^{\alpha_j, a_j} h(z) &= \sum_{n=1}^\infty \phi_n \frac{\prod_{j=1}^p \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^p \Gamma(a_j + \frac{n-1}{\alpha_j})} z^{n-1} \\ &= z^{-1} \sum_{n=1}^\infty \phi_n \frac{\prod_{j=1}^p \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^p \Gamma(a_j + \frac{n-1}{\alpha_j})} z^n, \end{aligned}$$

$$(\phi_1 := 1, z \in U, \alpha_j > 0),$$

where the multiplicity of the analytic function z^{-1} in $U\backslash\{0\}$, is removed by requiring $\log z$ to be real when $z > 0$. Then by (9), we obtain

$$\begin{aligned}
|\mathfrak{D}_z^{\alpha_j,a_j} h(z)| &\le \sum_{n=1}^{\infty} |\phi_n| \frac{\prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n-1}{\alpha_j})} r^{n-1} \\
&\le \sum_{n=1}^{\infty} n \frac{\prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n-1}{\alpha_j})} r^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{n}{n!} \frac{\prod_{j=1}^{p+1} \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n-1}{\alpha_j})} r^{n-1}, \ a_{p+1} = \alpha_{p+1} = 1 \\
&= \sum_{n=1}^{\infty} \frac{n}{n!} \frac{\prod_{j=1}^{p+1} \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^{p} \Gamma((a_j - \frac{1}{\alpha_j}) + \frac{n}{\alpha_j})} r^{n-1} \\
&= \left({}_{p+1}\Psi_p \left[\begin{array}{c} (a_1, \frac{1}{\alpha_1}), ..., (a_p, \frac{1}{\alpha_p}), (1,1); \\ \\ (a_1 - \frac{1}{\alpha_1}, \frac{1}{\alpha_1}), ..., (a_p - \frac{1}{\alpha_p}, \frac{1}{\alpha_p}); \end{array} |z| \right] \right)'.
\end{aligned}$$

□

Theorem 4.2. *Let the function h be in C (the class of convex functions). Then*

$$|\mathfrak{D}_z^{\alpha_j,a_j} h(z)| \le \left({}_{p+1}\Psi_p \left[\begin{array}{c} (a_1, \frac{1}{\alpha_1}), ..., (a_p, \frac{1}{\alpha_p}), (0,1); \\ \\ (a_1 - \frac{1}{\alpha_1}, \frac{1}{\alpha_1}), ..., (a_p - \frac{1}{\alpha_p}, \frac{1}{\alpha_p}); \end{array} |z| \right] \right)'.$$

Proof. Suppose that the function $h(z) \in C$ is given by $h(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$. Then in virtue of (10), we receive

$$\begin{aligned}
|\mathfrak{D}_z^{\alpha_j,a_j} h(z)| &\le \sum_{n=1}^{\infty} |\phi_n| \frac{\prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n-1}{\alpha_j})} r^{n-1} \\
&\le \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n-1}{\alpha_j})} r^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{n\Gamma(n)}{n!} \frac{\prod_{j=1}^{p+1} \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n-1}{\alpha_j})} r^{n-1} \\
&= \sum_{n=1}^{\infty} n \frac{\prod_{j=1}^{p+1} \Gamma(a_j + \frac{n}{\alpha_j})}{\prod_{j=1}^{p} \Gamma(a_j + \frac{n-1}{\alpha_j})} \frac{r^{n-1}}{n!}, \ a_{p+1} = 0, \ \alpha_{p+1} = 1 \\
&= \left({}_{p+1}\Psi_p \left[\begin{array}{c} (a_1, \frac{1}{\alpha_1}), ..., (a_p, \frac{1}{\alpha_p}), (0,1); \\ \\ (a_1 - \frac{1}{\alpha_1}, \frac{1}{\alpha_1}), ..., (a_p - \frac{1}{\alpha_p}, \frac{1}{\alpha_p}); \end{array} |z| \right] \right)'.
\end{aligned}$$

This completes the proof. □

5. Conclusion

We defined multi-order fractional operators (integral and differential) in the unit disk. We utilized these operators to generalize the Beer-Lambert equation. Solutions are imposed in terms of the multi-order Mittag-Leffler functions as well as the Wright functions. Furthermore, some geometric properties are discussed for the differential operator.

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